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The convergence of partial sums of interpolating polynomials

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Abstract

For functions of ΛBV, we study the convergence of the partial sums of interpolating polynomials. An estimate is found for the Fourier–Lagrange coefficients of these functions. For functions in BV, convergence is shown at points of discontinuity if the order of the polynomial increases sufficiently rapidly compared to the order of the partial sum. A Dirichlet–Jordan type theorem is shown for functions of harmonic bounded variation, and this result is shown to be best possible.

Keywords: Trigonometric interpolation; Lambda bounded variation; Partial sums of interpolating polynomials; Magnitude of coefficients

1. Introduction

Let f be a Riemann integrable function of period 2π, $t_0^{(n)}$ an arbitrary real number, and $t_j^{(n)} = t_0^{(n)} + \frac{2\pi j}{2n+1}$, $j = 0, \ldots, 2n$. Then $I_n(x, f)$ will denote the trigonometric polynomial which coincides with f at the $(2n+1)$-*fundamental points* $\{t_j^{(n)}\}$. If $D_n(t)$ denotes the nth Dirichlet kernel, $\sin(n + \frac{1}{2})t/2\sin\frac{1}{2}t$, then

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$$I_n(x,f)=\frac{2}{2n+1}\sum_{j=0}^{2n} f\bigl(t_j^{(n)}\bigr)D_n\bigl(x-t_j^{(n)}\bigr)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)\,d\omega_{2n+1}(t),$$

where $\omega_{2n+1}(t)$ is a step function with jumps $2\pi/(2n+1)$ at the points $t_j^{(n)}$, $j=0,\pm1,\pm2,\ldots.$ *Par abus de langage* we will refer to *this* set of points as the *fundamental points*. We can write

$$I_n(x,f)=\frac{1}{2}a_0+\sum_{\nu=0}^{n}\bigl(a_\nu^{(n)}\cos\nu x+b_\nu^{(n)}\sin\nu x\bigr),$$

where the Fourier–Lagrange coefficients $a_\nu^{(n)}$ and $b_\nu^{(n)}$ are given by

$$a_\nu^{(n)}=\frac{1}{\pi}\int_0^{2\pi} f(t)\cos\nu t\,d\omega_{2n+1}(t)\quad\text{and}\quad b_\nu^{(n)}=\frac{1}{\pi}\int_0^{2\pi} f(t)\sin\nu t\,d\omega_{2n+1}(t).$$

The complex Fourier–Lagrange coefficients $c_\nu^{(n)}$ are defined analogously. We shall be interested in the partial sums of the interpolating polynomials,

$$I_{n,\nu}(x,f)=\frac{1}{2}a_0+\sum_{k=0}^{\nu}\bigl(a_k^{(n)}\cos kx+b_k^{(n)}\sin kx\bigr)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_\nu(x-t)\,d\omega_{2n+1}(t).$$

We always assume that $n\geqslant\nu$.

Note that if T and T' are intervals of length 2π and ξ is a 2π-periodic function, then $\int_T\xi\,d\omega_{2n+1}(t)=\int_{T'}\xi\,d\omega_{2n+1}(t)$. So the integrals above may be taken over any interval of length 2π.

Zygmund [12, Chapter X, 5.4] has shown:

Proposition 1. *If f is of bounded variation, then $I_{n,\nu}(x,f)\to f(x)$ as $\nu\to\infty$, $n\geqslant\nu$, at every point of continuity of f. The convergence is uniform on every closed interval of points of continuity of f.*

The proof he gives relies on his estimate of the order of magnitude of the Fourier–Lagrange coefficients and the $(C,1)$-summability of $\{I_{n,\nu}\}$. He then applies a Tauberian theorem of Hardy to obtain the desired result.

We give an alternate proof of this result based directly on the definition of bounded variation. The method may be applied to analyze the convergence of the sequence $\{I_{n,\nu}(x,f)\}$ for functions of generalized bounded variation. In particular, it may be used for the class HBV, the functions of harmonic bounded variation and we will compare the convergence properties for functions of this class to those for functions in larger ΛBV classes.

We also obtain an estimate of the order of magnitude of the Fourier–Lagrange coefficients of functions of ΛBV which agrees with the known estimate when $\Lambda BV=BV$.

In his treatment of these problems, Zygmund often uses the modified Dirichlet kernel $D_\nu^*(t)=(\sin\nu t)/(2\tan t/2)$ to estimate $I_{n,\nu}$. This kernel is obtained by deleting one-half the last term in $I_{n,\nu}$. He does not use the kernel $\mathcal{D}_\nu(t)=\sin\nu t/t$. Estimating by use of this kernel makes many of our necessary computations more transparent and we prove its validity.

While, in general, we cannot reasonably expect convergence of the sequence of partial sums at points of discontinuity, we show that, for $f\in BV$, $I_{n,\nu}(x,f)\to f(x)$ at points of discontinuity if n increases sufficiently rapidly compared to ν.

2. Underlying definitions and known results

Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive real numbers such that $\sum 1/\lambda_n$ diverges. A function f defined on a real interval I is said to be of *Λ-bounded variation*, $f \in \Lambda BV$, if for every infinite collection of nonoverlapping intervals $I_n \subset I$, we have $\sum |f(I_n)|/\lambda_n < \infty$ or, equivalently, such sums have a common finite upper bound for all finite collections. The infimum of these upper bounds is the *Λ-variation, $V_\Lambda(f, I)$*. If $\Lambda = \{n\}$, we call this class the functions of *harmonic bounded variation*, *HBV*. It is known that the Dirichlet–Jordan theorem can be extended to functions of the class *HBV* and that this result is, in a sense, best possible. These notions were introduced and developed in [1–11], where all results that we use concerning these classes are to be found.

The *oscillation* of a function f on an interval I, $\sup\{|f(s) - f(t)|\colon s, t \in I\}$, will be denoted by $\operatorname{osc}(f, I)$.

Let Φ be a family of functions of period 2π. The functions $f \in \Phi$ are said to be *uniformly integrable R* if

(a) the functions are uniformly bounded;
(b) for every $\varepsilon > 0$ there is a $p_0 = p_0(\varepsilon)$ with the following property: for each $f \in \Phi$ we can find $p \leqslant p_0$ intervals $i_1, i_2, \dots, i_p$ in $(0, 2\pi)$ such that the oscillation of f over each i_k is less than ε and the set complementary to the i_k is of measure less than ε.

We have the following interesting result of Zygmund [12, Chapter X, 4.7], which is central in many of his arguments.

Proposition 2. *The Fourier–Lagrange coefficients $c_\nu^{(n)}$ tend uniformly to* 0 *as $|\nu| \to \infty$, $n \geqslant |\nu|$, for any set of functions f uniformly integrable R.*

We cannot use this result as stated when employing the kernel $\mathcal{D}_\nu(t)$, but we introduce an analogous result, stated as a lemma, which is of similar importance in our estimates.

3. Results

The following are our principal results.

Lemma 1. *Let $\{f(x,t)\}_x$ be a family of functions such that*

(i) *there is an $M < \infty$ such that $\|f(x,\cdot)\|_\infty < M$ for every x;*
(ii) *for every $\varepsilon > 0$ there is a $p_0 = p_0(\varepsilon)$ with the property: for each x there are $p \leqslant p_0$ intervals $I_1^x, \dots, I_p^x$ in $(x - \pi, x + \pi)$ such that, for each k, $\operatorname{osc}(f(x,\cdot), I_k^x) < \varepsilon$ and $|\bigcup_1^p I_k^x| > 2\pi - \varepsilon$.*

Then $\int_{x-\pi}^{x+\pi} f(x,t) e^{-i\nu t}\, d\omega_{2n+1}(t) \to 0$ as $|\nu| \to \infty$, $n \geqslant |\nu|$, uniformly in x.

Theorem 1. *$I_{n,\nu}(x, f) - \frac{1}{\pi} \int_{x-\pi}^{x+\pi} f(x+t) \mathcal{D}_\nu(t)\, d\omega_{2n+1}(t) = o(1)$ as $\nu \to \infty$, $n \geqslant \nu$, uniformly in x.*

Theorem 2. *For $f \in \Lambda BV$, $|c_{|\nu|}^{(n)}| \leqslant (\frac{1}{\sum_1^{2n+1} 1/\lambda_j}\} \frac{2n+1}{2|\nu|} V_\Lambda(f)$.*

Remark 1. If $\lambda_n = 1$, then $\Lambda BV = BV$ and we obtain Zygmund's estimate.

Theorem 3. *If $f \in HBV$, then $I_{n,\nu}(x, f) \to f(x)$ as $\nu \to \infty$, $n \geqslant \nu$, if f is continuous at x. Convergence is uniform on any closed interval of points of continuity.*

Theorem 4. *If $\Lambda BV \setminus HBV \neq \emptyset$, then there exists a continuous $f \in \Lambda BV$ such that $\{I_{n,\nu}(x, f)\}$ diverges at some point.*

Theorem 5. *If $f \in BV$, then $I_{n,\nu}(x, f) \to f(x)$ as $\nu \to \infty$, $n \geqslant \nu$, at points of discontinuity, if $\nu = o(\sqrt{n})$.*

4. Lemma

Proof. For a given ε and f, let

$$\overline{f}(x,t) = \sum_{k=1}^{p} \sup\left[f(x,\cdot), I_k^x\right]\chi_{I_k^x}(t).$$

Then

$$\begin{aligned}\int_{x-\pi}^{x+\pi} & f(x,t)e^{-i\nu t}\,d\omega_{2n+1}(t)\\ &= \int_{x-\pi}^{x+\pi} \overline{f}(x,t)e^{-i\nu t}\,d\omega_{2n+1}(t) - \int_{x-\pi}^{x+\pi} \left[f(x,t) - \overline{f}(x,t)\right]e^{-i\nu t}\,d\omega_{2n+1}(t)\\ &= I_1 + I_2.\end{aligned}$$

By computing the value of I_1 in the case where $\overline{f}$ is the characteristic function of a single interval, we see that

$$|I_1| \leqslant \frac{\pi p_0 M}{|\nu|}$$

independent of x, and so $I_1 \to 0$ as $|\nu| \to \infty$, $n \geqslant |\nu|$, uniformly in x. Now

$$|I_2| \leqslant \int_{x-\pi}^{x+\pi} \left|f(x,t) - \overline{f}(x,t)\right| d\omega_{2n+1}(t) = \int_{\bigcup I_k^x} \cdots + \int_{(x-\pi,x+\pi)\setminus\bigcup I_k^x} \cdots = I_2' + I_2''.$$

We have

$$\left|I_2'\right| \leqslant \varepsilon \int_{\bigcup I_k^x} d\omega_{2n+1}(t) \leqslant 2\pi\varepsilon$$

uniformly in x and

$$\left|I_2''\right| \leqslant M \int_{(x-\pi,x+\pi)\setminus\bigcup I_k^x} d\omega_{2n+1}(t) \to M\left|(x-\pi, x+\pi) \setminus \bigcup I_k^x\right| < M\varepsilon$$

as $\nu \to \infty$, $n \geqslant |\nu|$, uniformly in x. $\square$

5. Theorem 1

Proof. Let

$$g(x,t) = \frac{1}{2\tan\frac{1}{2}(t-x)} - \frac{1}{t-x}.$$

We wish to show that

$$\int_{x-\pi}^{x+\pi} f(t)g(x,t)\sin\nu(t-x)\,d\omega_{2n+1}(t) \to 0 \quad \text{as } \nu\to\infty,\ n\geqslant\nu,\ \text{uniformly in } x,$$

which will be a consequence of

$$\int_{x-\pi}^{x+\pi} f(t)g(x,t)e^{-i\nu t}\,d\omega_{2n+1}(t) \to 0 \quad \text{as } |\nu|\to\infty,\ n\geqslant|\nu|,\ \text{uniformly in } x.$$

Since f is bounded and g is continuous,

$$\bigl|f(t)g(x,t)\bigr| \leqslant \|f\|_\infty \|g\|_\infty.$$

Since f is 2π-periodic and Riemann integrable, given any $\varepsilon > 0$, there is $p > 0$ such that, in any interval of length 2π, there exist $I_1,\dots,I_p$, nonoverlapping intervals so that

$$\operatorname{osc}(f, I_k) < \varepsilon/2\|g\|_\infty \quad \text{and} \quad \Bigl|\bigcup I_k\Bigr| > 2\pi - \varepsilon.$$

Choose such intervals in $(x-\pi, x+\pi)$ and denote them by I_k^x. There is a $\delta > 0$ such that $I \subset (x-\pi, x+\pi)$ and

$$|I| < \delta \quad \text{implies } \operatorname{osc}\bigl(g(x,\cdot), I\bigr) < \varepsilon/2\|f\|_\infty.$$

Divide each of the intervals I_k^x into intervals of length less than δ to form a collection of intervals J_j^x, $j = 1,\dots,p_0$. Note that p_0 may be chosen to be independent of x. Then

$$\operatorname{osc}\bigl(f\cdot g, J_k^x\bigr) < \|f\|_\infty \cdot \varepsilon/2\|f\|_\infty + \|g\|_\infty \cdot \varepsilon/2\|g\|_\infty = \varepsilon \quad \text{and} \quad \Bigl|\bigcup J_k^x\Bigr| > 2\pi - \varepsilon.$$

Thus $\{f(t)g(x,t)\}_x$ is a set of functions satisfying the hypotheses of our lemma, from which the conclusion follows readily. □

6. Proposition 1

We turn now to the proof of Proposition 1, even though it is a corollary of Theorem 3, since its proof motivates the arguments needed for that of Theorem 3.

Proof. We will show that

$$I = \int_{x-\pi}^{x+\pi} \bigl[f(t) - f(x)\bigr]\mathcal{D}_\nu(t-x)\,d\omega_{2n+1}(t) \to 0$$

uniformly on any closed interval of points of continuity as $\nu\to\infty$, $n\geqslant\nu$. Write

$$I = \int_{x-\delta}^{x+\delta} \cdots + \int_{E_x} \cdots = I_1 + I_2, \quad \text{where } E_x = (x-\pi, x+\pi)\setminus(x-\delta, x+\delta).$$

We see that I_2 may be estimated by applying our lemma to the family

$$\left\{ f(t)\frac{\chi_{E_x}(t)}{t-x} \right\}_x.$$

Next we write $I_1 = \int_x^{x+\delta} \cdots + \int_{x-\delta}^{x} \cdots$, and note that we need consider only one of these integrals. We assume that $2\nu + 1 > 2\pi/\delta$. Let

$$h_\nu = \frac{2\pi}{2\nu+1}.$$

We can then write

$$\int_x^{x+\delta} \cdots = \int_x^{x+h_\nu} \cdots + \int_{x+h_\nu}^{x+\delta} \cdots,$$

and

$$\left| \int_x^{x+h_\nu} \big(f(x) - f(t)\big) \frac{\sin \nu(t-x)}{t-x}\, d\omega_{2n+1}(t) \right| \leqslant \sup_{t\in[x,x+h_\nu]} \big|f(t) - f(x)\big| \nu \left(\frac{h_\nu}{h_n} + 1 \right) h_n$$
$$= o(1)$$

as $\nu \to \infty$, $n \geqslant \nu$, uniformly on any closed interval of points of continuity.

When we have shown such a result for

$$\int_{x+h_\nu}^{x+\delta} \cdots = \sum_{i=p}^{q} \big[f(t_i) - f(x)\big] \mathcal{D}_\nu(t_i - x) h_n,$$

where $t_p, \ldots, t_q$ are the fundamental points in $[x + h_\nu, x + \delta]$, we shall be done.

Let $V(f, I)$ denote the variation of f over an interval I. Let $u_i = t_i - x$ and

$$d_{r,\nu} = \sum_{i=p}^{r} \mathcal{D}_\nu(u_i), \quad p \leqslant i \leqslant q \text{ and } d_{p-1,\nu} = 0;$$

then

$$\mathcal{D}_\nu(u_i) = d_{i,\nu} - d_{i-1,\nu}.$$

Note. If $\{a_i\}$ is a nonincreasing sequence of positive numbers and $B_i = b_p + \cdots + b_i$, $p \leqslant i \leqslant q$, then $|\sum_p^q a_i b_i| \leqslant a_p \max_i |B_i|$. Letting $a_i = 1/u_i$ and $b_i = \sin \nu u_i$, we see that

$$|d_{r,\nu}| \leqslant \frac{1}{u_p} \max_{i\leqslant q} \left| \sum_{j=p}^{i} \sin \nu u_j \right|.$$

Now

$$\sin \nu u_j \sin \nu h_n = \frac{1}{2}\big[\cos \nu(u_j - h_n) - \cos \nu(u_j + h_n)\big]$$
$$= \frac{1}{2}\big[\cos \nu u_{j-1} - \cos \nu u_{j+1}\big],$$

so that

$$\left|\sum_{j=p}^{i} \sin \nu u_j \sin \nu h_n\right| = \frac{1}{2}\left|\sum_{j=p}^{i}\bigl[\cos \nu u_{j-1} - \cos \nu u_{j+1}\bigr]\right|$$
$$= \frac{1}{2}|\cos \nu u_{p-1} + \cos \nu u_p - \cos \nu u_i - \cos \nu u_{i+1}|$$
$$\leqslant 2.$$

Thus

$$|d_{r,\nu}| \leqslant \frac{2}{u_p}\left|\frac{1}{\sin \nu h_n}\right| \leqslant \frac{2}{h_\nu}\frac{1}{\frac{2}{\pi}\nu h_n} < \frac{2}{h_n},$$

and we have

$$\left|\int_{x+h_\nu}^{x+\delta} \cdots\right| = h_n\left|\sum_{i=p}^{q}\bigl[f(t_i) - f(x)\bigr](d_{i,\nu} - d_{i-1,\nu})\right|$$
$$= h_n\left|\sum_{i=p}^{q-1}\bigl[f(x+u_i) - f(x+u_{i+1})\bigr]d_{i,\nu} + \bigl[f(x+u_q) - f(x)\bigr]d_{q,\nu}\right.$$
$$\left. + \bigl[f(x+u_p) - f(x)\bigr]d_{p-1,\nu}\right|$$
$$\leqslant h_n \sum_{i=p}^{q-1}\bigl|f(x+u_i) - f(x+u_{i+1})\bigr||d_{i,\nu}| + \bigl|f(x+u_q) - f(x)\bigr||d_{q,\nu}|$$
$$< h_n \frac{2}{h_n}\bigl[V\bigl(f,(x,x+\delta)\bigr) + \bigl|f(x+u_q) - f(x)\bigr|\bigr].$$

Given $\varepsilon > 0$, we can choose $\delta > 0$ so that $|f(t) - f(x)| < \varepsilon/4$ for t in $[x, x+\delta]$ and $V(f,(x,x+\delta)) < \varepsilon/4$ if f is continuous at x. If f is continuous at each point of a closed interval, then δ can be chosen so that these estimates are uniform for x in that interval. □

7. Theorem 2

Proof. There is no loss of generality if we choose fundamental points $t_j = jh_n$, $j = 0, \pm 1, \pm 2, \ldots$. Let

$$S_k = \sum_{j=0}^{k} e^{-i\nu t_j}, \quad k = 0, \ldots, 2n, \quad \text{and} \quad S_{-1} = 0.$$

Note that $S_{2n} = 0$, since $e^{-i\nu h_n}$ is a root of $1 - x^{2n+1} = 0$, and

$$|S_k| = \left|\frac{1 - e^{-i\nu(k+1)h_n}}{1 - e^{-i\nu h_n}}\right| \leqslant \frac{2}{|1 - e^{-i\nu h_n}|} = \frac{2}{\sqrt{2 - 2\cos \nu h_n}} = \frac{1}{\sin \frac{\pi|\nu|}{2n+1}} \leqslant \frac{2n+1}{2|\nu|}.$$

If T is any interval of length 2π, then

$$c_\nu^{(n)} = \frac{1}{2\pi}\int_T f(t)e^{-i\nu t}\, d\omega_{2n+1}(t).$$

Let $T=[jh_n, jh_n+2\pi)$, $j=1,2,\dots,2n+1$. Then

$$\begin{aligned} c_\nu^{(n)} &= \frac{1}{2n+1}\sum_{k=0}^{2n} f(t_k+jh_n)e^{-i\nu(t_k+jh_n)} \\ &= \frac{1}{2n+1}e^{-ijh_n}\sum_{k=0}^{2n} f(t_k+jh_n)(S_k-S_{k-1}) \\ &= \frac{e^{-ijh_n}}{2n+1}\sum_{k=0}^{2n-1}\bigl[f(t_k+jh_n)-f(t_{k+1}+jh_n)\bigr]S_k. \end{aligned}$$

Using our estimate of S_k,

$$\bigl|c_\nu^{(n)}\bigr|/\lambda_j \leqslant \frac{1}{2n+1}\frac{2n+1}{2|\nu|}\sum_{k=0}^{2n-1}\bigl|f(t_k+jh_n)-f(t_{k+1}+jh_n)\bigr|/\lambda_j,$$

so we have

$$\begin{aligned} \bigl|c_\nu^{(n)}\bigr|\sum_{j=1}^{2n+1}1/\lambda_j &\leqslant \frac{1}{2|\nu|}\sum_{j=1}^{2n+1}\sum_{k=0}^{2n-1}\bigl|f(t_k+jh_n)-f(t_{k+1}+jh_n)\bigr|/\lambda_j \\ &= \frac{1}{2|\nu|}\sum_{k=0}^{2n-1}\sum_{j=1}^{2n+1}\bigl|f(t_k+jh_n)-f(t_{k+1}+jh_n)\bigr|/\lambda_j \\ &\leqslant \frac{2n+1}{2|\nu|}V_\Lambda\bigl(f,(0,2\pi)\bigr), \end{aligned}$$

which is the desired result. □

8. Theorem 3

Proof. We are now assuming that $f\in HBV$. Let $V_H(f,I)$ denote the harmonic variation of f on I. As was the case for BV, an application of our lemma reduces the problem to estimating

$$\int_{x+h_\nu}^{x+\delta}\bigl[f(t)-f(x)\bigr]\mathcal{D}_\nu(t-x)\,d\omega_{2n+1}(t).$$

There is no loss of generality in assuming that the fundamental points are jh_n, $j=0,\pm1,\pm2,\dots.$ Let $p=[h_\nu/h_n]$ and $s=[\delta/ph_n]$, where $[\xi]$ denotes the greatest integer less than or equal to ξ. Then $ph_n\leqslant h_\nu\leqslant(p+1)$, implying

$$ph_n\to 0\quad\text{and}\quad s\to\infty\quad\text{as } \nu\to\infty.$$

Let

$$x_i^{(j)}=x+(jp+i)h_n\quad\text{and}\quad t_i^{(j)}=x_i^{(j)}-x.$$

Then

$$\int_{x+h_\nu}^{x+\delta}\cdots=\int_{x+h_\nu}^{x_p^{(s-1)}}\cdots+\int_{x_p^{(s-1)}}^{x+\delta}\cdots=A+B.$$

Let

$$d_{0,i}=0 \quad \text{and} \quad d_{j,i}=\sum_{r=1}^{j}\sin\nu t_i^{(r)}, \quad \text{for } i=1,\dots,p \text{ and } j=1,\dots,s-1.$$

Then, for $j\geqslant 1$,

$$\begin{aligned}
d_{j,i}\sin\left(\frac{1}{2}\nu p h_n\right) &= \sum_{r=1}^{j}\sin\nu(rp+i)h_n\sin\left(\frac{1}{2}\nu p h_n\right)\\
&= \frac{1}{2}\sum_{r=1}^{j}\left[\cos\nu\left(\left(r-\frac{1}{2}\right)p+i\right)h_n-\cos\nu\left(\left(r+\frac{1}{2}\right)p+i\right)h_n\right]\\
&= \frac{1}{2}\left[\cos\nu(p/2+i)h_n-\cos\nu\left(\left(j+\frac{1}{2}\right)p+i\right)h_n\right],
\end{aligned}$$

implying

$$|d_{j,i}|\leqslant\frac{1}{|\sin\frac{1}{2}\nu p h_n|}.$$

Now

$$0<\frac{1}{2}\nu p h_n\leqslant\frac{\nu}{2}\frac{2n+1}{2\nu+1}\frac{2\pi}{2n+1}=\frac{\nu\pi}{2\nu+1}<\frac{\pi}{2},$$

and, writing $h_\nu/h_n=p+\eta$, $0\leqslant\eta<1$,

$$\frac{1}{2}\nu p h_n=\frac{\nu}{2}\frac{p}{p+\eta}\frac{2\pi 0}{2\nu+1}>\frac{\pi}{6},$$

implying

$$|d_{j,i}|<2.$$

Now

$$\begin{aligned}
A &= h_n\sum_{i=1}^{p}\sum_{j=1}^{s-1}\frac{f(x_i^{(j)})-f(x)}{t_i^{(j)}}\sin\nu t_i^{(j)}\\
&= h_n\sum_{i=1}^{p}\sum_{j=1}^{s-1}\frac{f(x_i^{(j)})-f(x)}{t_i^{(j)}}(d_{j,i}-d_{j,i-1})\\
&= h_n\sum_{i=1}^{p}\sum_{j=1}^{s-2}\left[\frac{f(x_i^{(j)})-f(x)}{t_i^{(j)}}-\frac{f(x_i^{(j+1)})-f(x)}{t_i^{(j+1)}}\right]d_{j,i}\\
&\quad+h_n\sum_{i=1}^{p}\frac{f(x_i^{(s-1)})-f(x)}{t_i^{(s-1)}}d_{s-1,i}\\
&= I+II.
\end{aligned}$$

Then

$$
\begin{aligned}
I &= h_n \sum_{i=1}^{p} \sum_{j=1}^{s-2} \frac{f(x_i^{(j)}) - f(x_i^{(j+1)})}{t_i^{(j)}} d_{j,i} \\
&\quad + h_n \sum_{i=1}^{p} \sum_{j=1}^{s-2} \bigl[f\bigl(x_i^{(j+1)}\bigr) - f(x) \bigr] \biggl(\frac{1}{t_i^{(j)}} - \frac{1}{t_i^{(j+1)}} \biggr) d_{j,i} \\
&= I_1 + I_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
|I_1| &\leqslant 2 \sum_{i=1}^{p} \sum_{j=1}^{s-2} \frac{|f(x_i^{(j)}) - f(x_i^{(j+1)})|}{jp+i} \leqslant \frac{2}{p} \sum_{i=1}^{p} \sum_{j=1}^{s-2} \frac{|f(x_i^{(j)}) - f(x_i^{(j+1)})|}{j} \\
&\leqslant 2 V_H\bigl(f, (x, x+\delta)\bigr).
\end{aligned}
$$

Given $\varepsilon > 0$, we may choose $\delta > 0$ so that $|f(t) - f(x)| < \varepsilon/2$ for $t \in (x, x+\delta)$, and this is uniform for x in a closed interval of points of continuity. Then

$$
|I_2| \leqslant 2 h_n \varepsilon \sum_{i=1}^{p} \sum_{j=1}^{s-2} \biggl(\frac{1}{t_i^{(j)}} - \frac{1}{t_i^{(j+1)}} \biggr),
$$

and

$$
\sum_{j=1}^{s-2} \biggl(\frac{1}{t_i^{(j)}} - \frac{1}{t_i^{(j+1)}} \biggr) = \frac{1}{t_i^{(1)}} - \frac{1}{t_i^{(s-1)}} < \frac{1}{t_i^{(1)}} < \frac{1}{p h_n},
$$

so

$$
|I_2| < \frac{\varepsilon}{2} 2 p h_n \frac{1}{p h_n} = \varepsilon.
$$

Note that

$$
x_p^{(s-1)} = x + s p h_n > x + \delta - p h_n \geqslant x + \delta - h_\nu.
$$

Thus

$$
\begin{aligned}
|B| &\leqslant \sup_{t \in (x, x+\delta)} \bigl| f(t) - f(x) \bigr| \int_{x+\delta-h_\nu}^{x+\delta} \frac{1}{t-x} \, d\omega_{2n+1}(t) \\
&< \varepsilon \int_{x+\delta-h_\nu-h_n}^{x+\delta+h_n} \frac{1}{t-x} \, dt = \varepsilon \ln \frac{\delta + h_n}{\delta - h_\nu - h_n} = \varepsilon o(1)
\end{aligned}
$$

as $\nu \to \infty$, an estimate which holds uniformly on a closed interval of points of continuity, completes the proof of Theorem 3. $\square$

9. Theorem 4

Proof. The condition $\Lambda BV \setminus HBV \neq \emptyset$ implies that there exists a sequence $\{a_n\} \searrow 0$, such that $\sum a_n/\lambda_n$ converges and $\sum a_n/n$ diverges. Let our fundamental points be

$$
-\pi + j h_n, \quad j = 0, \pm 1, \pm 2, \ldots .
$$

We define $f_n(x)$ as follows:

$$f_n(x) = 0, \quad x = \frac{(4k-1)}{2n+1}\pi, \quad k = 0, 1, 2, \ldots, \left[\frac{n+1}{2}\right],$$
$$f_n(x) = a_k, \quad x = \frac{(4k+1)}{2n+1}\pi, \quad k = 0, 1, 2, \ldots, \left[\frac{n-1}{2}\right].$$

Note that

$$\frac{4[\frac{n+1}{2}]-1}{2n+1}\pi = \begin{cases} \pi & \text{for odd } n, \\ \frac{2n-1}{2n+1}\pi & \text{for even } n. \end{cases}$$

We have defined $f(x)$ at the endpoints of intervals of length $2\pi/(2n+1)$ beginning at $-\pi/(2n+1)$ and ending at $(4[\frac{n+1}{2}]-1)\pi/(2n+1)$. Let f be linear on each of these closed intervals, equal to 0 on the rest of $[-\pi, \pi]$ and extended with period 2π. Now ΛBV and ΛBV_C, the space of continuous functions in ΛBV, are Banach spaces. The functions f_n are in ΛBV_C and

$$\begin{aligned} \|f_n\|_\Lambda &= \left|f_n(-\pi)\right| + V_\Lambda\left(f_n, [0, 2\pi]\right) \\ &= \begin{cases} a_1/\lambda_1 + a_1/\lambda_2 + a_2/\lambda_3 + a_2/\lambda_4 + \cdots + a_{(n+1)/2}/\lambda_{n+1}, & n \text{ odd}, \\ a_1/\lambda_1 + a_1/\lambda_2 + a_2/\lambda_3 + a_2/\lambda_4 + \cdots + a_{n/2}/\lambda_n, & n \text{ even}, \end{cases} \\ &< 2\sum_1^\infty \frac{a_k}{\lambda_k} < \infty. \end{aligned}$$

Let

$$T_n(f_n) = I_n(0, f_n).$$

Then, assuming n to be even,

$$\begin{aligned} T_n(f_n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,d\omega_{2n+1}(t) \\ &= \frac{1}{2n+1}\sum_{k=0}^{2n} f_n\left(t_k^{(n)}\right)\frac{\sin(n+\frac{1}{2})t_k^{(n)}}{\sin\frac{1}{2}t_k^{(n)}} \\ &= \frac{1}{2n+1}\sum_{k=0}^{n/2-1} a_{k+1}\frac{\sin(2k\pi+\pi/2)}{\sin\frac{\pi}{2}\frac{1+4k}{2n+1}} \\ &> \frac{1}{2n+1}\sum_{k=0}^{n/2-1}\frac{a_{k+1}}{\frac{\pi}{2}\frac{1+4k}{2n+1}} \\ &= \frac{2}{\pi}\sum_{k=0}^{n/2-1}\frac{a_{k+1}}{1+4k} > \frac{1}{2\pi}\sum_{k=0}^{n/2-1}\frac{a_{k+1}}{1+k} = \frac{1}{2\pi}\sum_{k=1}^{n/2}\frac{a_k}{k} \to \infty \end{aligned}$$

as $n \to \infty$. For n odd, the result is the same. Thus we have

$$\|T_n\| \geqslant \frac{|T_n(f_n)|}{\|f_n\|_\Lambda} \to \infty$$

as $n \to \infty$, and the Banach–Steinhaus theorem implies that there is a continuous $f \in \Lambda BV$ such that $\{I_n(0, f)\}$ does not converge. □

10. Theorem 5

Proof. Let x be point of discontinuity of $f \in BV$. Then the partial sums of the Fourier series of f at x, $S_\nu(x, f)$, converge to $[f(x+) + f(x-)]/2$ as $\nu \to \infty$.

Consider the sine coefficients of $I_{n,\nu}$ and S_ν. Note that

$$\int_0^{2\pi} f(t) \sin kt \, d\omega_{2n+1}$$

is a Riemann sum approximating

$$\int_0^{2\pi} f(t) \sin kt \, dt.$$

Thus if $E_j = (\frac{2\pi j}{2n+1}, \frac{2\pi(j+1)}{2n+1})$,

$$\begin{aligned}
\left|b_k^{(n)} - b_k\right| &\leqslant \frac{1}{\pi} \frac{2\pi}{2n+1} \sum_{j=0}^{2n} \operatorname{osc}\bigl(f(t) \sin kt, E_j\bigr) \\
&\leqslant \frac{2}{2n+1} \sum_{j=0}^{2n} \bigl[\|f\|_\infty \operatorname{osc}(\sin kt, E_j) + \|\sin kt\|_\infty \operatorname{osc}\bigl(f(t), E_j\bigr)\bigr] \\
&\leqslant \frac{2}{2n+1} \bigl[\|f\|_\infty V\bigl(\sin kt, (0, 2\pi)\bigr) + \|\sin kt\|_\infty V\bigl(f(t), (0, 2\pi)\bigr)\bigr] \\
&= \frac{2}{2n+1} \bigl[\|f\|_\infty 2k + V\bigl(f(t), (0, 2\pi)\bigr)\bigr].
\end{aligned}$$

A similar estimate holds for $|a_k^{(n)} - a_k|$, $k = 1, \ldots, n$, and

$$\left|a_0^{(n)} - a_0\right|/2 \leqslant \frac{1}{2n+1} V\bigl(f, (0, 2\pi)\bigr).$$

Thus

$$\left|S_\nu(x, f) - I_{n,\nu}(x, f)\right| \leqslant \frac{4\nu+1}{2n+1} V\bigl(f(t), (0, 2\pi)\bigr) + 2\|f\|_\infty \frac{2\nu(\nu+1)}{2n+1}$$

and $\nu = o(\sqrt{n})$ implies that $I_{n,\nu}(x, f)$ converges to $[f(x+) + f(x-)]/2$. □

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